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# The second Hopf map and Yang–Coulomb system on a 5D (pseudo)sphere

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## Abstract

Using the second Hopf map, we perform the reduction of the eight-dimensional (pseudo)spherical (Higgs)oscillator to a five-dimensional system interacting with a Yang monopole. Then, using a standard trick, we obtain, from the latter system, the pseudospherical and spherical generalizations of the Yang–Coulomb system (the five-dimensional analog of MICZ-Kepler system). We present the whole set of its constants of motions, including the hidden symmetry generators given by the analog of the Runge–Lenz vector. In the same way, starting from the eight-dimensional anisotropic inharmonic Higgs oscillator, we construct the integrable (pseudo)spherical generalization of the Yang–Coulomb system with the Stark term.

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## 1. Introduction

It is well-known that in some cases the hidden symmetries of the oscillator and the Coulomb system can be related. More precisely, both in the classical and quantum cases, the transformation  $r = R^2$  converts the  $(p + 1)$ -dimensional radial Coulomb problem to a  $2p$ -dimensional radial oscillator ( $r$  and  $R$  denote the radial coordinates of Coulomb and oscillator systems, respectively). The angular parts of these systems, which are  $(2p - 1)$ - and  $p$ -dimensional spheres, can be related to each other in the distinguished cases  $p = 1, 2, 4, 8$ , due to the existence of the Hopf maps  $S^{2p-1}/S^p = S^{p-1}$  for  $p = 1, 2, 4, 8$  (see the review [1] and refs therein). Hence, in these cases, one can establish a complete correspondence between the Coulomb and the oscillator systems. For the first three cases this correspondence has been established many years ago. The corresponding transformations are known in the literature as Bohlin (or Lévi-Civita) [2], Kustaanheimo–Stiefel [3] and Hurwitz [4] transformations. These transformations imply the reduction of the oscillator system by an action of the group  $Z_2, U(1)$ ,

$SU(2)$  and yield, indeed, to systems which are more general than the Coulomb one; these systems inherit the Coulomb symmetry but are specified by the presence of a monopole. In the case of the  $Z_2$  group ( $p = 1$ ) it is the two-dimensional Coulomb problem with spin  $1/2$  anyon (magnetic flux) [5]; in the case of the  $U(1)$  group ( $p = 2$ ) it is the so-called MICZ-Kepler system, the generalization of the three-dimensional Coulomb system in the presence of a Dirac monopole [6]; in the case of the  $SU(2)$  group ( $p = 4$ ) it is the so-called Yang–Coulomb (or  $SU(2)$ -Kepler) system, the generalization of the five-dimensional Coulomb system in the presence of a Yang monopole [7].

On the other hand, the oscillator and Coulomb systems admit generalizations to a  $d$ -dimensional sphere and a two-sheet hyperboloid (pseudosphere) with a radius  $R_0$  given by the potentials [8, 9]

$$V_{\text{osc}} = \frac{\omega^2}{2} \frac{\mathbf{x}^2}{x_{d+1}^2}, \quad V_C = -\frac{\gamma}{|\mathbf{x}|} \frac{x_{d+1}}{x_{d+1}^2}, \quad (1.1)$$

where  $\mathbf{x}, x_{d+1}$  are the (pseudo)Euclidean coordinates of the ambient space  $\mathbb{R}^{d+1}(\mathbb{R}^{d,1})$ :  $\epsilon \mathbf{x}^2 + x_{d+1}^2 = R_0^2$ ,  $\epsilon = \pm 1$ . Here  $\epsilon = +1$  corresponds to the sphere and  $\epsilon = -1$  corresponds to the pseudosphere. These systems possess nonlinear hidden symmetries providing them with properties similar to those of the conventional oscillator and Coulomb systems. Detailed considerations of these systems can be found in [10] and refs therein. The relation between these systems has been established in [11] for the cases  $p = 1, 2$  only, though it was clear from the considerations there, that it could be straightforwardly extended to the higher dimensional case  $p = 4$ . Let us note that both the oscillators on the sphere and pseudosphere result, upon the mentioned reduction, in Coulomb-like systems on the *pseudosphere*. For example, the  $p = 2$  case yields a system with Coulomb symmetry which can be identified with the pseudospherical MICZ-Kepler system. Then, after an obvious ‘Wick rotation’, it can be mapped in the spherical MICZ-Kepler system suggested in [12]. Moreover, in this way, starting from the appropriate anisotropic inharmonic four-dimensional (pseudo)spherical Higgs oscillator, one can obtain the integrable (pseudo)spherical generalization with the Stark term [13]. Hence, the extension of the (pseudo)spherical oscillator–Coulomb correspondence to the  $p = 4$  case is an important task; it should give us the integrable generalizations of the five-dimensional Yang–Coulomb system, including the systems with the Stark term. The solution of this task is the purpose of our paper. We will use the procedure of the Lagrangian  $SU(2)$ -reduction of eight-dimensional system considered in our recent paper [14] in the context of supersymmetric mechanics. We will restrict ourself to classical considerations only. The (pseudo)spherical Yang–Coulomb system, obtained in the present paper, besides the presence of a Yang monopole, is specified by the presence of a specific centrifugal term

$$\Delta U = \frac{s^2}{2g(r)r^2}, \quad (1.2)$$

where  $g(r) dx^\mu dx^\mu$  is the conformal invariant metric on the (pseudo)sphere and  $s^2$  is the square of the isospin of the system. Upon quantization it should be replaced by  $\hbar^2 s(s + 1)$ ,  $s = 0, \pm 1/2, \pm 1, \dots$ . In [17] it was demonstrated that the five-dimensional rotationally invariant system of the Yang monopole, supplied by the addition of the above potential, preserves the analytic form of the energy spectrum. The only change in the spectrum of the system is the range of validity of the orbital quantum number. Its lower bound shifts from zero to  $|s|$ . Hence, one can immediately write down the spectrum of the (pseudo)spherical Yang–Coulomb system, taking into account general statements. But why are we so sure that the quantum mechanical reduction of the eight-dimensional Higgs oscillator to the (pseudo)spherical Yang–Coulomb system should lead to a system with such spectrum

(in general, reduction and quantization are not commuting procedures)? We refer to the second reference in [11], where the consistency of the quantum mechanical spectrum of the eight-dimensional Higgs oscillator and pseudospherical Yang–Coulomb system has been demonstrated. Hence, for the unperturbed quantum-mechanical systems everything works finely. With the quantum mechanics of the (pseudo)spherical Yang–Coulomb systems with the Stark term the situation is more complicated. In contrast with the spherically symmetric systems, the impact of the monopole is less trivial. Even in the three-dimensional planar case, the presence of a (Dirac) monopole essentially changes the spectrum of the MICZ-Kepler system with the Stark term (and of its modification) obtained within perturbation theory [18]. Similar calculations for the (pseudo)spherical case yield technical complications, which we hope to address in future studies.

The paper is arranged as follows. In the second section we present the explicit description of the second Hopf map in terms needed for our purposes and employ it to reduce the eight-dimensional bosonic system to a lower dimensional system with  $SU(2)$  monopole. In the third section we apply the previous construction to the oscillator on eight-dimensional (pseudo)sphere and get, from the reduced system, the five-dimensional (pseudo)spherical generalization of Yang–Coulomb system. In a similar way we obtain the five-dimensional (pseudo)spherical generalization of Yang–Coulomb system with the Stark term.

## 2. $SU(2)$ reduction and the second Hopf map

For the description of the second Hopf map  $S^7/S^3 = S^4$  we first introduce five  $8 \times 8$  matrices  $\Gamma^\mu$

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\delta^{\mu\nu} \mathbf{1}_8 \quad (2.1)$$

with the following relations:

$$\begin{aligned} \Gamma^1 &= \tau_A \otimes \tau_1 \otimes \tau_A, & \Gamma^2 &= \tau_A \otimes \tau_2 \otimes \tau_A, & \Gamma^3 &= \tau_A \otimes \tau_A \otimes \mathbf{1}_2, \\ \Gamma^4 &= \tau_1 \otimes \mathbf{1}_2 \otimes \mathbf{1}_2, & \Gamma^5 &= \tau_2 \otimes \mathbf{1}_2 \otimes \mathbf{1}_2, \end{aligned} \quad (2.2)$$

where

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \tau_A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{1}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (2.3)$$

where  $\{A, B\}$  denotes the anticommutator. For our purposes we have also to introduce three  $8 \times 8$  antisymmetric matrices  $\Sigma_a$ :

$$\Sigma^1 = \frac{1}{2} \mathbf{1}_2 \otimes \tau_A \otimes \tau_1, \quad \Sigma^2 = \frac{1}{2} \mathbf{1}_2 \otimes \tau_A \otimes \tau_2, \quad \Sigma^3 = \frac{1}{2} \mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \tau_A, \quad (2.4)$$

which commute with all matrices  $\Gamma_\mu$ , anticommute with each other and satisfy the  $SU(2)$  algebra relations:

$$[\Gamma^\mu, \Sigma^i] = 0, \quad \{\Sigma^i, \Sigma^j\} = -2\delta^{ij} \mathbf{1}_8, \quad [\Sigma^i, \Sigma^j] = \varepsilon_{ijk} \Sigma^k. \quad (2.5)$$

Let us now have an eight-dimensional conformal flat space with metric  $g$ , parametrized by eight coordinates  $u_A$ . We also consider 5 functions  $x_\mu$ , which are connected with  $u_i$  by the following relations:

$$x^\mu = u^T \Gamma^\mu u, \quad \mu = 1, \dots, 5 \quad (2.6)$$

where  $u$  is an eightdimensional column vector with elements  $u_A$ .

One can note that the transformation

$$u \rightarrow (\lambda_0 \mathbf{1}_8 + \lambda_i \Sigma_i) u, \quad \lambda_0^2 + \sum \lambda_i^2 = 1, \quad i = 1, 2, 3 \quad (2.7)$$

leaves invariant the  $x_\mu$  quantities. Here and further, summation is understood whenever repeated indices appear. Therefore, fibration 2.6 identifies all points that are mapped into each other by transformation 2.7. It can be checked explicitly that

$$x_\mu x_\mu \equiv r^2 = (u_A u_A)^2 \equiv R^4. \quad (2.8)$$

Thus, defining the seven-dimensional sphere in  $\mathbb{R}^8$  of radius  $R$ :  $\mathbf{u}_\alpha \bar{\mathbf{u}}_\alpha = R^2$ , we get a four-dimensional sphere in  $\mathbb{R}^5$  with radius  $r = R^2$ , i.e. we obtain the second Hopf map. Taking into account relation 2.8 and the fact that the second relation in 2.7 defines the  $S^3$  sphere, one can conclude that the second Hopf map is a fibration of the sphere  $S^7$  over  $S^3$ :

$$S^7/S^3 = S^4.$$

Let us parametrize the bundle  $S^3 = S^2 \times S^1$  by the following coordinates:

$$z = \frac{u_7 - \mathbf{i}u_8}{u_5 - \mathbf{i}u_6}, \quad \bar{z} = \frac{u_7 + \mathbf{i}u_8}{u_5 + \mathbf{i}u_6}, \quad \gamma = \arctan \frac{u_5}{u_6}, \quad (2.9)$$

where the coordinates  $z, \bar{z}$  parametrize  $S^2 = CP^1$  and  $\gamma$  parametrizes  $S^1$ .

The matrices  $\Sigma^i$  define a set of vector-fields on  $S^3$  that form the  $SU(2)$  algebra:

$$\mathbf{U}_i = u_A \Sigma_{AB}^i \frac{\partial}{\partial u_B}. \quad (2.10)$$

In terms of the new coordinates these vector-fields can be written as follows:

$$\mathbf{U}_3 = -\frac{1}{2} \frac{\partial}{\partial \gamma}, \quad \mathbf{U}_2 + \mathbf{i}\mathbf{U}_1 = \mathbf{U}_+ = \frac{e^{-2i\gamma}}{4} \left( (1 + z\bar{z}) \frac{\partial}{\partial \bar{z}} + \frac{iz}{2} \frac{\partial}{\partial \gamma} \right), \quad \mathbf{U}_- = \bar{\mathbf{U}}_+. \quad (2.11)$$

The one-forms dual to this set of vector-fields look as follows:

$$\tilde{\Lambda}_3 = 2 d\gamma + \mathbf{i} \frac{z d\bar{z} - \bar{z} dz}{1 + z\bar{z}}, \quad \tilde{\Lambda}_+ = \tilde{\Lambda}_2 + \mathbf{i}\tilde{\Lambda}_1 = 2 \frac{e^{2i\gamma} d\bar{z}}{1 + z\bar{z}} \quad (2.12)$$

$$\tilde{\Lambda}_3(\mathbf{U}_3) = \tilde{\Lambda}_\pm(\mathbf{U}_\pm) = 1, \quad \tilde{\Lambda}_\pm(\mathbf{U}_\mp) = \tilde{\Lambda}_\pm(\mathbf{U}_3) = \tilde{\Lambda}_3(\mathbf{U}_\pm) = 0. \quad (2.13)$$

We will also need another set of  $SU(2)$  Lie algebra elements parametrizing the sphere  $S^3$  and commuting with (2.11)

$$\mathbf{V}_3 = \frac{1}{2} \frac{\partial}{\partial \gamma} + \iota \left( z \frac{\partial}{\partial z} - \bar{z} \frac{\partial}{\partial \bar{z}} \right), \quad \mathbf{V}_+ = \frac{1}{2} \left( \frac{\partial}{\partial \bar{z}} + z^2 \frac{\partial}{\partial z} - \iota \frac{z}{2} \frac{\partial}{\partial \gamma} \right), \quad \mathbf{V}_- = \bar{\mathbf{V}}_+. \quad (2.14)$$

The one-forms dual to these vector-fields look as follows:

$$\Lambda_3 = 2h_3 d\gamma + \iota \frac{\bar{z} dz - z d\bar{z}}{1 + z\bar{z}}, \quad \Lambda_+ = \Lambda_2 + \mathbf{i}\Lambda_1 = 2\mathbf{i}h_+ d\gamma + 2 \frac{d\bar{z}}{1 + z\bar{z}}, \quad (2.15)$$

$$\Lambda_3(\mathbf{V}_3) = \Lambda_\pm(\mathbf{V}_\pm) = 1, \quad \Lambda_\pm(\mathbf{V}_\mp) = \Lambda_\pm(\mathbf{V}_3) = \Lambda_3(\mathbf{V}_\pm) = 0, \quad (2.16)$$

here  $h_3, \mathbf{h}_\pm$  are the Cartesian coordinates of the ambient space  $\mathbb{R}^3$ , defining the Killing potentials of  $S^2$

$$\mathbf{h}_+ = \frac{2z}{1 + \bar{z}z}, \quad h_{p+1} = \frac{1 - \bar{z}z}{1 + \bar{z}z}. \quad (2.17)$$

It can be checked directly that the vector fields  $U$  and  $V$  commute with each other. Precisely, this pair forms the the  $so(4) = s(3) \times so(3)$  algebra of isometries of the  $S^3$  sphere.

$$[\mathbf{V}_i, \mathbf{V}_j] = \varepsilon_{ijk} \mathbf{V}_k, \quad [\mathbf{U}_i, \mathbf{U}_j] = \varepsilon_{ijk} \mathbf{U}_k, \quad [\mathbf{V}_i, \mathbf{U}_j] = 0. \quad (2.18)$$

Let us consider now the particle on the eight-dimensional space equipped with the  $SU(2)$ -invariant conformal flat metric moving. In a specific potential that depends only on the coordinates  $x_\mu$ . In the next section a physical example of this Lagrangian will be considered.

$$\mathcal{L}_8 = g\dot{u}_A\dot{u}_A + U, \tag{2.19}$$

where  $g$  and  $U$  depend on functions  $x_\mu = u^T \Gamma^\mu u$  only, so that they are invariant under the action of the  $SU(2)$  group. Since we are interested in the construction of spherical generalization of the MICZ-Kepler systems, we have chosen the conformal flat metric case. The generalization to nondiagonal metric (e.g. IHIP<sup>2</sup>) is straightforward. Various metrics and potentials are considered in [16].

In the new parametrization the Lagrangian of that particle has the following form:

$$\mathcal{L}_8 = g\dot{r}_\mu\dot{r}_\mu + \frac{gr}{2}\Lambda_i A_i - \frac{gr}{4}\Lambda_i \Lambda_i + U, \tag{2.20}$$

where

$$r_\kappa = \frac{x_\kappa}{\sqrt{2(r+x_5)}}, \quad \kappa \leq 4 \quad \text{and} \quad r_5 = \sqrt{\frac{r+x_5}{2}}.$$

Here and further  $\Lambda_i$  is defined by (2.15), where the differentials are replaced by the respective time derivatives, while

$$A_i = A_{\beta\dot{x}\beta}^i = \frac{\eta_{\alpha\beta}^i x_\alpha \dot{x}_\beta}{r(r+x_5)}, \quad \eta_{\alpha\beta}^i = \delta_{i\alpha}\delta_{4\beta} - \delta_{4\alpha}\delta_{i\beta} + \varepsilon_{4i\alpha\beta},$$

$$i = 1, 2, 3, \quad \alpha, \beta = 1, 2, 3, 4 \tag{2.21}$$

$\eta_{bc}^i$  is t'Hooft symbol.

It can be seen that  $A_a$  defines the potential of the  $SU(2)$  Yang monopole [15]. Below we will show that after reduction by  $SU(2)$  group action this term will describe the physical coupling of a Yang monopole to the system.

By use of the Noether constants of motion, we can decrease the dimensionality of the system. Due to the non-Abelian nature of the  $SU(2)$  group, the system will be reduced to a  $(5 + 1)$ -dimensional one.

For the correct reduction procedure we have to replace the initial Lagrangian by a variationally equivalent one, extending the initial configuration space by the new variables  $\pi, \bar{\pi}, p_\gamma$  which play the role of momenta conjugated to  $z, \bar{z}, \gamma$ . In other words, we will replace the sphere  $S^3$  (parametrized by  $z, \bar{z}, \gamma$ ), by its cotangent bundle  $T^*S^3$  parametrized by coordinates  $z, \bar{z}, \gamma, \pi, \bar{\pi}, p_\gamma$ . Let us define, on  $T^*S^3$ , the Poisson brackets given by the relations

$$\{\pi, z\} = 1, \quad \{\bar{\pi}, \bar{z}\} = 1, \quad \{p_\gamma, \gamma\} = 1. \tag{2.22}$$

We introduce the Hamiltonian generators  $P_a$  corresponding to the vector fields (2.14) (replacing the derivatives in vector fields  $\mathbf{V}_a$  by corresponding momenta)

$$P_+ = \frac{P_2 - \iota P_1}{2} = \frac{1}{2} \left( \pi + \bar{z}^2 \bar{\pi} - \frac{\iota \bar{z}}{2} p_\gamma \right), \quad P_- = \bar{P}_-, \quad P_3 = \frac{p_\gamma}{2} + \iota(z\pi - \bar{z}\bar{\pi}). \tag{2.23}$$

In the same way we introduce the Hamiltonian generators  $I_a$  corresponding to the vector fields (2.11):

$$I_3 = -\frac{p_\gamma}{2}, \quad I_+ = \frac{I_2 - \iota I_1}{2} = \frac{\iota p_\gamma z + 2\bar{\pi}(1+z\bar{z})}{4} e^{-2\iota\gamma}, \quad I_- = \bar{I}_+. \tag{2.24}$$

These quantities define, with respect to the Poisson brackets (2.22), the  $so(4) = so(3) \times so(3)$  algebra

$$\{P_i, P_j\} = \varepsilon_{ijk} P_k, \quad \{I_i, I_j\} = \varepsilon_{ijk} I_k, \quad \{I_i, P_j\} = 0. \tag{2.25}$$

The functions  $P_i, I_i$  obey the following equality, which is important for our considerations:

$$I_i I_i = P_i P_i. \tag{2.26}$$

We replace now the initial Lagrangian (2.20) by the following variationally equivalent one:

$$\mathcal{L}_{\text{int}} = (P_+ \Lambda_+ + P_- \Lambda_- + P_3 \Lambda_3) + (\pi_\mu - P_i A_\mu^i) \dot{x}_\mu - \frac{P_i P_i}{gr} - r \frac{\pi_\mu \pi_\mu}{g} + U(x). \tag{2.27}$$

Here we used the identity

$$-\frac{gr}{4} A_i A_i + g \dot{r}_\mu \dot{r}_\mu = g \frac{\dot{x}_\mu \dot{x}_\mu}{4r}.$$

Such kind of representation of a variationally equivalent Lagrangian is motivated by the following reason. Let the  $2n$ -dimensional Lagrangian have the form

$$L = f_r(y) \dot{y}^r - H(y), \quad r = 1 \dots 2n, \tag{2.28}$$

where  $H(y)$  and  $f_\mu(y)$  are some functions of the variable  $y$ . The Euler–Lagrange equations for such Lagrangian look as

$$\dot{y}^r = \omega^{rs} \frac{\partial H}{\partial y^s}, \tag{2.29}$$

where  $\omega^{rs} \omega_{sq} = \delta_q^r$ ,  $\omega_{rq} = \partial_r f_q - \partial_q f_r$ . It is easy to see that  $\omega^{rq}$  defines Poisson Brackets  $\{y^r, y^s\} = \omega^{rs}$ . Hence,  $\omega_{rq}$  is the symplectic structure of the system and  $H$  is the corresponding Hamiltonian. In the invariant language we represent the Lagrangian as follows:

$$L = \omega^{(1)}(dy) - H(y), \tag{2.30}$$

where  $\omega^{(1)}$  is one-form which locally can be written as in (2.28), and the symplectic structure looks as follows:

$$\omega^{(2)} = d\omega^{(1)}. \tag{2.31}$$

We will use this fact in the next section.

The isometries of this, new, Lagrangian corresponding to (2.11) are defined by the vector fields

$$\tilde{U}_i \equiv \{I_i, \quad \}, \tag{2.32}$$

where  $I_i$  are given by (2.24) and the Poisson brackets are given by (2.22). Indeed,  $I_i$  are precisely the Noether constants of motion of the new Lagrangian (2.27) corresponding to (2.32). To see this we simply should take into account the following equality:

$$P_+ \Lambda_+ + P_- \Lambda_- + P_3 \Lambda_3 = p_\gamma \dot{\gamma} + \pi \dot{z} + \bar{\pi} \dot{\bar{z}}. \tag{2.33}$$

Let us perform now the reduction by the action of the  $SU(2)$  group given by the vector fields (2.32). For this purpose we have to fix the Noether constants of motion (2.24):

$$I_i = s_i = \text{const}, \quad s_i s_i \equiv s^2.$$

Since  $I_i$  are constants of motion independent of the  $r_\mu$  coordinates we can perform an orthogonal rotation, so that only the third component of this set,  $I_3$ , will be different from zero. Equating  $I_+$  and  $I_-$  with zero we obtain

$$-I_3 = \frac{p_\gamma}{2} = s, \quad \bar{\pi} = -s i \frac{z}{1+z\bar{z}}, \quad \pi = s i \frac{\bar{z}}{1+z\bar{z}}. \tag{2.34}$$

Hence,

$$P_+ = -s \frac{i z}{1+z\bar{z}}, \quad P_- = s \frac{i \bar{z}}{1+z\bar{z}}, \quad P_3 = s \frac{1-z\bar{z}}{1+z\bar{z}}. \tag{2.35}$$

Thus,  $P_a$  become precisely the Killing potentials of the  $S^2$  sphere! It is not an occasional coincidence, indeed.

Taking in mind the equality (2.33) we conclude that the third term in (2.27) can be ignored because it is a full time derivative. Also, taking into account (2.26) and denoting  $\tilde{g} = g/2r$ , one can rewrite the Lagrangian (variationally equivalent) as follows:

$$\mathcal{L}_{\text{int}}^{\text{red}} = (\pi_\mu - sA_\mu^i h_i) \dot{x}_\mu - \iota s \frac{\bar{z}\dot{z} - z\dot{\bar{z}}}{1 + z\bar{z}} - \frac{\pi_\mu \pi_\mu}{2\tilde{g}} - \frac{s^2}{2\tilde{g}r} - U(\mathbf{x}), \quad \mu = 1, \dots, 5. \quad (2.36)$$

According to the definition, the reduced Lagrangian can be obtained after performing a variation procedure in terms of the variables  $\pi_\mu$ . The second term in this reduced Lagrangian is the one-form defining the symplectic (and Kähler) structure on  $S^2$ , in agreement with the above observation that  $P_a$  results in the Killing potentials of  $S^2$ .

Thus, the Noether constants of motion do not allow us to exclude the  $z, \bar{z}$  variables. However, their time derivatives appear in the Lagrangian in a linear way only, defining the internal degrees of freedom of the five-dimensional isospin particle interacting with the Yang monopole. As a consequence, the dimensionality of the phase space of the reduced system is  $2 \cdot 5 + 2 = 12$ . Only in the particular case  $s = 0$ , corresponding to the absence of Yang monopole, we arrive at a five-dimensional system. Hence, *locally*, the Lagrangian of the system can be formulated in the six-dimensional space. We will use this fact in the next section.

### 3. Higgs oscillator and (pseudo)spherical Yang–Coulomb system

Let us apply the above construction to the Higgs oscillator on the eight-dimensional sphere and pseudosphere and obtain, for the reduced system, the (pseudo)spherical generalization of the Yang–Coulomb system, in the spirit of [11].

For this purpose we introduce the conformal flat coordinates of  $d$ -dimensional (pseudo)sphere, which are precisely the stereographic coordinates. These coordinates are related to the Cartesian coordinates of the ambient  $(d + 1)$ -dimensional space as follows (here and in the following we assume the unit radius of the sphere and pseudosphere):

$$x_A = \frac{2u_A}{1 + \epsilon u^2}, \quad x_{d+1} = \frac{1 - \epsilon u^2}{1 + \epsilon u^2}, \quad A = 1, \dots, d, \quad (3.1)$$

where  $u^2 = u^T u$ . Here  $x_A, x_{d+1}$  are the (pseudo)Euclidean coordinates of the ambient space  $\mathbb{R}^{d+1}(\mathbb{R}^{d,1})$ :  $\epsilon x_A^2 + x_{d+1}^2 = 1, \epsilon = \pm 1$ . The  $\epsilon = +1$  corresponds to the sphere and  $\epsilon = -1$  corresponds to the pseudosphere.

In these coordinates the metric takes the conformally flat form

$$ds^2 = \frac{4 du_A du_A}{(1 + \epsilon u^2)^2}, \quad (3.2)$$

while the potentials of the Higgs oscillator and of the Schrödinger–Kepler system (1.1) read

$$V_{\text{osc}} = \frac{2\omega^2 u^2}{(1 - \epsilon u^2)^2}, \quad V_C = -\gamma \frac{1 - \epsilon u^2}{2|u|}. \quad (3.3)$$

Hence, the  $SU(2)$ -reduction of the the Higgs oscillator on the (pseudo)sphere to a 5D system leads to the following Lagrangian (compare with 2.36):

$$\mathcal{L}_{\text{osc}}^{\text{red}} = \mathcal{L}_{\text{red}} = (\pi_\mu - sA_\mu^i h_i) \dot{x}_\mu - \iota s \frac{\bar{z}\dot{z} - z\dot{\bar{z}}}{1 + z\bar{z}} - \frac{(1 + \epsilon r)^2}{4} r \left( \frac{\pi_\mu \pi_\mu}{2} + \frac{s^2}{2r^2} \right) - \frac{2\omega^2 r}{(1 - \epsilon r)^2}. \quad (3.4)$$



The one-form corresponding to this Lagrangian has the following form:

$$\omega^{(1)} = \pi_\mu dx_\mu - s A_\mu^i h_i dx_\mu + \iota s \left( \frac{\bar{z} dz - z d\bar{z}}{1 + z\bar{z}} \right), \quad (3.5)$$

and the inverse matrix of corresponding symplectic structure  $\omega^{(2)} = d\omega^{(1)}$  defines the Poisson brackets

$$\begin{aligned} \{\pi_\mu, \pi_\nu\} &= s (\partial_\mu A_\nu^i - \partial_\nu A_\mu^i - \varepsilon_{ijk} A_\mu^j A_\nu^k) h_i \equiv s F_{\mu\nu}^i h_i, & \{z, \bar{z}\} &= \frac{\iota}{2s} (1 + z\bar{z})^2, \\ \{h_i, h_j\} &= \frac{1}{s} \varepsilon_{ijk} h_k, & \{\pi_\mu, x_\nu\} &= \delta_{\mu\nu}. \end{aligned} \quad (3.6)$$

The Hamiltonian of the reduced system is given by the expression

$$\mathcal{H}_{\text{red}}^{\text{osc}} = \frac{(1 + \epsilon r)^2}{4} r \left( \frac{\pi_\mu \pi_\mu}{2} + \frac{s^2}{2r^2} \right) + \frac{2\omega^2 r}{(1 - \epsilon r)^2}. \quad (3.7)$$

On the other hand, the Higgs oscillator has a number of symmetries: besides the rotational  $so(8)$  symmetries, defining the Noether constants of motion, it possesses constants of motion which are quadratic in momenta. We are interested in their  $SU(2)$  invariant subset given by the generators

$$J_{\mu\nu} = \frac{\mathcal{P}^T [\Gamma_\mu, \Gamma_\nu] u}{2} \quad (3.8)$$

and

$$\mathbf{A} = \frac{J^T \Gamma_\mu J}{2} + 2 \frac{\omega^2 u^T \Gamma_\mu u}{(1 - \epsilon u^2)^2}, \quad (3.9)$$

where  $J_A = (1 - \epsilon u^2) \mathcal{P}_A + 2\epsilon(\mathbf{u}\mathcal{P})u_A$  and  $\mathcal{P}_A$  is the corresponding momenta of the coordinate  $u_A$ .

Reducing the generators of rotations to the 5D system, following the general procedure described in the previous section, we get

$$J_{\mu\nu} = x_\mu \pi_\nu - x_\nu \pi_\mu + 2r^2 F_{\mu\nu}^i h_i. \quad (3.10)$$

In order to find the expressions for the hidden symmetry generators we exclude the subset of the generators of rotations that leaves invariant the coordinate  $x_\mu$ :

$$\mathcal{J}_{\alpha\beta}^\mu = \varepsilon_{\mu\alpha\beta\nu\lambda} J_{\nu\lambda}, \quad \alpha, \beta = 1 \dots 4, \quad \mu, \nu, \lambda = 1 \dots 5. \quad (3.11)$$

Now, we can write the implicit expression for  $A_\mu$  in the following form:

$$A_\mu = \frac{J_{\mu\nu} T_\nu}{2} + \frac{q_\mu}{4} (1 + \epsilon r)^2 \left( \pi_\mu \pi_\mu + \frac{s^2}{r^2} \right) + \frac{2\omega^2 q_\mu}{(1 - \epsilon r)^2} + \frac{\epsilon}{2} \varepsilon_{\alpha\beta\gamma\delta} \mathcal{J}_{\alpha\beta}^\mu \mathcal{J}_{\gamma\delta}^\mu, \quad (3.12)$$

where  $T_\mu = (1 + q^2) p_\mu - 2(\mathbf{q}\mathbf{p})q_\mu$ - denote transition operators on the pseudosphere. Since the last terms are expressed through the already mentioned motion integrals, we can ignore them.

Following [11], we can now transform the reduced Higgs oscillator to a Kepler-like system. For this purpose we should fix the energy surface  $\mathcal{H}_{\text{osc}}^{\text{red}} = E \equiv \gamma/2$  and multiply by  $(1 - \epsilon r)^2/r$ , to get

$$(\mathcal{H}_{\text{osc}}^{\text{red}} - E_{\text{osc}}^{\text{red}}) \frac{(1 - \epsilon r)^2}{r} = 0 \equiv \mathcal{H}_{\text{MICZ}} - E_{\text{MICZ}}, \quad E_{\text{MICZ}} = -\epsilon\gamma - 2\omega^2 \quad (3.13)$$

$$\mathcal{H}_{\text{MICZ}} = \frac{(1 - r^2)^2}{4} \frac{1}{2} \left( \pi_\mu \pi_\mu + \frac{s^2}{r^2} \right) + \gamma \frac{1 + r^2}{2r}. \quad (3.14)$$

For any motion integral  $\mathcal{I}$  we have

$$\{H_{\text{MICZ}}, \mathcal{I}\} = \left\{ \left( H_{\text{osc}}^{\text{red}} - E_{\text{osc}}^{\text{red}} \right) \frac{(1 - \epsilon r)^2}{r}, \mathcal{I} \right\} \Big|_{H_{\text{osc}}^{\text{red}} = E_{\text{osc}}^{\text{red}}} = 0. \tag{3.15}$$

Hence, the new Hamiltonian has the same number of motion integrals and therefore preserves the integrability of the initial system. After fixing the energy surface, the quantities  $A_\mu$  transform, respectively, in the Runge–Lenz vector of the constructed  $SU(2)$ -Kepler (or Yang–Coulomb) system on pseudosphere

$$A_\mu = \frac{J_{\mu\nu} T_\nu}{2} + \gamma \frac{q_\mu}{r}. \tag{3.16}$$

Thus, we constructed the five-dimensional pseudospherical generalization of the MICZ–Kepler system, i.e. the pseudospherical Yang–Coulomb system, and found its constants of motion. It is not difficult to find a spherical analog of this system. Performing the Wick rotation, we obtain the spherical Yang–Coulomb system. It will be defined with the same Poisson brackets as before by the Hamiltonian

$$\mathcal{H}_{\text{MICZ}}^{(\text{sph})} = \frac{(1 + r^2)^2}{4} \frac{1}{2} \left( \pi_\mu \pi_\mu + \frac{s^2}{r^2} \right) + \gamma \frac{1 - r^2}{2r}, \tag{3.17}$$

and by the motion integrals  $A_\mu$ , where the quantities  $T_\mu$  are replaced by  $T_\mu = (1 - q^2)p_\mu + 2(\mathbf{q}\mathbf{p})q_\mu$ .

### 3.1. The anisotropic inharmonic oscillator and the Yang–Coulomb–Stark system

The oscillator described in the previous section can be extended by adding anisotropic and inharmonic parts [13].

$$\mathcal{L}_{\text{AIOSC}} = \frac{4}{(1 + \epsilon u^2)^2} \frac{\dot{u}_i \dot{u}_i}{2} - \frac{2\omega^2 u^2}{(1 - \epsilon u^2)^2} - \frac{2\Delta\omega^2 u \Gamma_5 u}{(1 - \epsilon u^2)^2} - \frac{4\epsilon_{el}}{(1 - (u^2)^2)^2} \frac{1 + (u^2)^2}{(1 - \epsilon u^2)^2} u^2 (u \Gamma_5 u). \tag{3.18}$$

Since the additional terms in this Lagrangian do not preserve the spherical symmetry of the previous system, only a part of the integrals of motion will be generalized. So, instead of  $N = 5(5 - 1)/2 = 10$  motion integrals corresponding to the rotation symmetry  $J_{\mu\nu}$ , we have only  $N' = 4(4 - 1)/2 = 6$  ones,  $J_{\alpha\beta}^5$ , defined in (3.11). Only one component of the generator of hidden symmetry is generalized. In the Hamiltonian approach it has the following form:

$$A_5 = \frac{J \Gamma_5 J}{2} + 2 \frac{\omega^2 u \Gamma_5 u}{(1 - \epsilon \bar{u} u)^2} + 2 \frac{\Delta\omega^2 u^2}{(1 + \epsilon \bar{u} u)^2} + 4\epsilon_{el} \left( \frac{(u^2)^2}{(1 - (u^2)^2)^2} + \frac{(u \Gamma_5 u)^2}{(1 - \epsilon u^2)^4} \right). \tag{3.19}$$

It is obvious that the expression for this quantity in the Lagrangian approach can be obtained just by replacing the momenta by the corresponding time derivative divided by  $(1 + \epsilon u^2)^2$ . In the same way as we obtained the MICZ–Kepler system in the previous section, we get now

$$\mathcal{H}_{\text{MICZ}} = \frac{(1 - r^2)^2}{4} \frac{1}{2} \left( \pi_\mu \pi_\mu + \frac{s^2}{r^2} \right) + \gamma \frac{1 + r^2}{2r} + 2\Delta\omega^2 \left( \frac{1 - \epsilon r}{1 + \epsilon r} \right)^2 \frac{x_5}{r} + 2\epsilon_{el} \frac{1 + r^2}{1 - r^2} \frac{x_5}{1 - r^2} \tag{3.20}$$

$$A_5 = \frac{J_{5\nu} T_\nu}{2} + \gamma \frac{x_5}{r} + 2 \frac{\Delta\omega^2}{(1 + \epsilon r)^2} \frac{r^2 - x_5^2}{r} + 2\epsilon_{el} \frac{r^2 - x_5^2}{(1 - r^2)^2}. \tag{3.21}$$

The fourth term in the Hamiltonian (3.20) in the limit of flat space yields a homogeneous electric field with strength  $\varepsilon_{el}$ . Hence, we can consider it as a generalization of the Stark term in the case of a (pseudo)-spherical space. The third term is just the  $\cos \theta$  potential.

As in the previous section, the transition to the sphere can be realized by performing a Wick rotation. All terms in (3.20) result in real expressions, except the third one. However, one can note that we can consider the real and imaginary parts of the Hamiltonian separately. Indeed, let the Hamiltonian  $\mathcal{H}$  and a motion integral  $\mathcal{I}$  have the following form:

$$\mathcal{H} = K_H + U + \mathbf{i}V, \quad \mathcal{I} = K_I + P + \mathbf{i}Q,$$

where  $U, V, P, Q$  are the real functions of coordinates and  $K_H$  and  $K_I$  are the kinetic terms of Hamiltonian and motion integral, respectively. The condition  $\{\mathcal{H}, \mathcal{I}\} = 0$  leads us to two equalities:

$$\{\mathcal{H}_{\text{Re}}, \mathcal{I}_{\text{Re}}\} = 0, \quad \{\mathcal{H}_{\text{Im}}, \mathcal{I}_{\text{Im}}\} = 0,$$

where  $\mathcal{H}_{\text{Re}} = K_H + U, \mathcal{I}_{\text{Re}} = K_I + P$  and  $\mathcal{H}_{\text{Im}} = K_H + V, \mathcal{I}_{\text{Im}} = K_I + Q$ . This still will not lead us to a new system. After separating the expression, we will find that its imaginary part looks exactly like the Stark term and therefore can be ignored. Explicitly we find

$$\begin{aligned} \mathcal{H}_{\text{MICZ}}^{\text{sph}} = & \frac{(1+r^2)^2}{4} \frac{1}{2} \left( \pi_\mu \pi_\mu + \frac{s^2}{r^2} \right) + \gamma \frac{1+r^2}{2r} \\ & + 2 \text{Re} \Delta \omega^2 \left( \frac{1-\iota \epsilon r}{1+\iota \epsilon r} \right)^2 \frac{x_5}{r} + 2\varepsilon_{el} \frac{1-r^2}{1+r^2} \frac{x_5}{1-r^2}. \end{aligned} \quad (3.22)$$

#### 4. Conclusion and discussion

In this paper we applied the  $SU(2)$  reduction procedure to the Higgs oscillator on the eight-dimensional pseudosphere and sphere, and transforming the energy level of the reduced system, we obtained the Yang–Coulomb systems on the five-dimensional sphere and pseudosphere. We recall that the Yang–Coulomb system is the generalization of the five-dimensional Coulomb system specified by the presence of  $SU(2)$  Yang monopole, which inherits the symmetries of the five-dimensional Coulomb system. Similarly, the constructed (pseudo)spherical Yang–Coulomb system inherits all the symmetries of the five-dimensional (pseudo)spherical Coulomb system. We also applied this procedure to the anisotropic inharmonic Higgs oscillator [13] and obtained, in this way, the integrable (pseudo)spherical generalization of the Yang–Coulomb system with the Stark term. While the spectrum of the previous system can be easily obtained from the general construction [17], the spectrum (even perturbative) of the latter one needs to be constructed. Even for the planar case it is unknown up to now. We are planning to investigate it in forthcoming studies.

There are related problems that definitely need to be studied. A few years ago the analog of oscillator on complex projective space  $CP^N$  has been suggested, which respected the inclusion of a constant magnetic field [19]. It was found that in the  $CP^2$  case this oscillator can be reduced to the (three-dimensional) MICZ-Kepler system on (pseudo)sphere. It is interesting to clarify whether the oscillator on  $CP^4$  results, upon  $SU(2)$  reduction, into the (pseudo)spherical Yang–Coulomb system. Also, in complete similarity to  $CP^N$ , one can define the oscillator on the quaternionic projective space  $HP^N$  respecting the inclusion of the instanton field [20]. One can expect that the  $SU(2)$  reduction of such an oscillator on  $HP^2$  would also yield a (pseudo)spherical Yang–Coulomb system. Both these statements should be carefully checked.

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